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## ABSTRACT

The concept of the  $k$ -Steiner interval is a natural generalization of the geodesic (binary) interval. It is defined as a mapping  $S : V \times \dots \times V \rightarrow 2^V$  such that  $S(u_1, \dots, u_k)$  consists of all vertices in  $G$  that lie on some Steiner tree with respect to a multiset  $W = \{u_1, \dots, u_k\}$  of vertices from  $G$ . In this paper we obtain, for each  $k$ , a characterization of the class of graphs in which every  $k$ -Steiner interval  $S$  has the so-called union property, which says that  $S(u_1, \dots, u_k)$  coincides with the union of geodesic intervals  $I(u_i, u_j)$  between all pairs from  $W$ . It turns out that, as soon as  $k > 3$ , this class coincides with the class of graphs in which the  $k$ -Steiner interval enjoys the monotone axiom (m), respectively (b2) axiom, the conditions from betweenness theory. Notably,  $S$  satisfies (m), if  $x_1, \dots, x_k \in S(u_1, \dots, u_k)$  implies  $S(x_1, \dots, x_k) \subseteq S(u_1, \dots, u_k)$ , and  $S$  satisfies (b2) if  $x \in S(u_1, u_2, \dots, u_k)$  implies  $S(x, u_2, \dots, u_k) \subseteq S(u_1, \dots, u_k)$ . In the case  $k = 3$ , these three classes are different, and we give structural characterizations of graphs for which their Steiner interval  $S$  satisfies the union property as well as the monotone axiom (m). We also prove several partial observations on the class of graphs in which the 3-Steiner interval satisfies (b2), which lead to the conjecture that these are precisely the graphs in which every block is a geodetic graph with diameter at most two.

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## 1. Introduction and preliminaries

For vertices  $u, v$  of a graph  $G$ , the interval  $I(u, v)$  in graphs is usually defined as the set of vertices lying on a *geodesic* (shortest path) between  $u$  and  $v$ . But besides geodesics there are some other notions that can be used for defining an interval, such as induced and detour paths. In this paper we will focus on yet another concept, the Steiner interval, and consider its relation with geodesic intervals.

The Steiner tree problem is a well-known problem with several variations and applications. It can concern points in Euclidean (or other metric) spaces, and vertices of weighted or non-weighted graphs [11], and has drawn much attention

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due to the development of approximation algorithms, see [1,16] and the references therein. In a non-weighted connected graph  $G$ , a *Steiner tree* of a (multi)set  $W \subseteq V(G)$ , is a minimum order tree in  $G$  that contains all vertices of  $W$ . The number of edges in a Steiner tree  $T$  of  $W$  is called the *Steiner distance* of  $W$ , denoted  $d(W)$ , while the *size* of  $T$  describes the number of vertices in  $T$  (i.e.  $d(W) + 1$ ). The  $k$ -*Steiner interval* is a mapping  $S : V \times \cdots \times V \rightarrow V$  such that  $S(u_1, u_2, \dots, u_k)$  consists of all vertices in  $G$  that lie on some Steiner tree with respect to  $\{u_1, \dots, u_k\}$ , where  $u_1, \dots, u_k$  are, not necessarily distinct, vertices of  $G$  (in this way  $S$  is an extension of  $I$ , as  $S(u, v, \dots, v) = I(u, v)$ ). (Note that, as above, we will simplify the notation for  $S(\{u_1, \dots, u_k\})$  to  $S(u_1, \dots, u_k)$ , where  $S$  denotes the  $k$ -Steiner interval, and  $u_1, \dots, u_k$  are not necessarily distinct vertices of a graph.) Steiner intervals on ordinary vertex subsets have been studied in several papers [6,8–11,13,16,20–22].

One of the main issues regarding Steiner intervals is related to connections between different variations of the geodetic number, see the survey paper [5]. Chartrand and Zhang proposed a natural concept of the Steiner number of a graph [6], and among several nice results “proved” an erroneous statement [6] regarding the connection between Steiner intervals of a set of vertices, and the union of geodesic intervals between pairs of the vertices from the set. This error was observed and corrected by Pelayo [22], and the development intrigued Hernando et al. [10] to raise the following problem: for which graphs the Steiner interval of any set of vertices, whose union of geodesic intervals between pairs of vertices in the set is the whole vertex set, also yields all vertices? Certainly this property holds for graphs in which  $S(W) \subseteq \bigcup_{u,v \in W} I(u, v)$  for all  $W \subseteq V(G)$ , which was shown to be true in distance hereditary graphs [10,21] and in a more general family of 3-Steiner distance hereditary graphs [8]. A characterization of these graphs (in which  $S(W) \subseteq \bigcup_{u,v \in W} I(u, v)$  for all  $W \subseteq V(G)$ ) remains open, and seems to be quite difficult. In this paper we consider the following stronger condition: given a fixed  $k$ , for any multiset  $W$  of vertices with  $|W| = k$ ,

$$S(W) = \bigcup_{u,v \in W} I(u, v).$$

We call this the *union property* of the  $k$ -Steiner interval. When  $k = 2$  the union property trivially holds in all graphs. We prove in Section 2 that for any  $k$  greater than 3, the union property holds precisely in block graphs. The case  $k = 3$  turns out to be the most difficult and interesting, see Section 3. (See also [9,13,20] for other studies on Steiner intervals.)

The second focus of this paper is on the concept of betweenness in graphs as introduced by Mulder in [18] (he implicitly considered this notion already in his book [17]). Starting points of his study were two very strong properties that the geodesic interval  $I$  enjoys. Namely, (i) if  $x$  is between  $u$  and  $v$  (i.e.  $x \in I(u, v)$ ) and  $x \neq u$ , then  $u$  is not between  $x$  and  $v$ , and, (ii) if  $x$  is between  $u$  and  $v$ , and  $y$  is between  $u$  and  $x$ , then  $y$  is between  $u$  and  $v$ . Properties (i) and (ii) are usually denoted by (b1) and (b2), respectively, and together they form *betweenness axioms* as defined in [18]. The interpretation of the betweenness properties in this sense was first studied by Mulder and Morgana [15], where it was also proved that the induced path interval  $J$  is a betweenness if and only if  $G$  is a house, hole, domino-free graph. A related property (not always satisfied by  $I$ ) is that if  $x$  and  $y$  are between  $u$  and  $v$ , and  $z$  is between  $x$  and  $y$ , then  $z$  is between  $u$  and  $v$ . This property is known as the *monotone axiom*, which is again introduced formally in [18]. (In [26], van de Vel uses the term *monotone law* for what we call the (b2) axiom.) The graphs in which the monotone axiom is always satisfied are known as *interval monotone graphs* which were also introduced in [17], see also [2,14]. Clearly the monotone axiom always implies (b2), but the converse need not hold and the characterization of interval monotone graphs is still an open problem. Betweenness in discrete structures other than graphs has been studied much earlier, for example see [24].

As 2-Steiner intervals are precisely the geodesic intervals  $I$ ,  $k$ -Steiner intervals form a generalization of the geodesic interval, hence it is natural to look at the analogous concept of betweenness for  $k$ -Steiner intervals. The betweenness axioms and the monotone axiom (m) can be generalized in a natural way from binary to  $k$ -ary functions (in particular from geodesic intervals to  $k$ -Steiner intervals) as follows: for any  $u_1, u_2, \dots, u_k, x, x_1, x_2, \dots, x_k \in V(G)$  which are not necessarily distinct,

- (b1)  $x \in S(u_1, u_2, \dots, u_k), x \neq u_1 \Rightarrow u_1 \notin S(x, u_2, \dots, u_k)$ ,
- (b2)  $x \in S(u_1, u_2, \dots, u_k) \Rightarrow S(x, u_2, \dots, u_k) \subseteq S(u_1, u_2, \dots, u_k)$ ,
- (m)  $x_1, x_2, \dots, x_k \in S(u_1, u_2, \dots, u_k) \Rightarrow S(x_1, x_2, \dots, x_k) \subseteq S(u_1, u_2, \dots, u_k)$ .

Somewhat surprisingly for the  $k$ -Steiner interval, where  $k > 2$ , the betweenness axioms are not satisfied in all graphs. As we show in Section 3, in the case  $k = 3$  the class of graphs in which the 3-Steiner interval has the union property (which are the graphs in which each block is a clique or a 5-cycle) is properly contained in the class of graphs in which the 3-Steiner interval satisfies the monotone axiom (m), which is in turn properly contained in the class of graphs in which the 3-Steiner interval satisfies (b2). Example of graphs, for which the monotone axiom is satisfied for the 3-Steiner interval  $S$  but not the union property are the graphs  $M_k$ ,  $k \geq 3$ , see Fig. 8. An example of a graph for which the 3-Steiner interval  $S$  satisfies (b2), but not (m) is the famous Petersen graph, see Fig. 10. One can easily verify that (m) is not satisfied by the Petersen graph. By using the labeling of vertices from Fig. 10, note that  $S(b, d, f)$  consists of all vertices in the graph except for  $w$ , yet  $w \in S(x, y, z)$ . Hence  $S(x, y, z) \not\subseteq S(b, d, f)$ , and (m) is not satisfied for  $S$ . On the other hand, for any  $k$  greater than 3 the classes of graphs in which the  $k$ -Steiner interval satisfies the union property, the monotone axiom, and the (b2) axiom are all the same, which is the main theorem in Section 2.

We conclude this section with the following lemma that considerably reduces the class of graphs in which the 3-Steiner interval satisfies the union property. This property is even more restrictive for the  $k$ -Steiner interval where  $k > 3$ . Recall that a subgraph  $H$  of a graph  $G$  is an *isometric subgraph* of  $G$  if for any pair of vertices  $u, v \in V(H)$ , there exists a geodesic

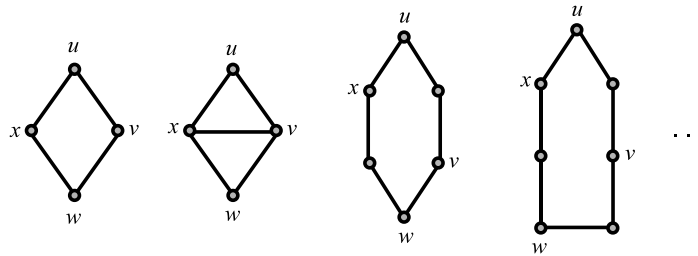


Fig. 1. Forbidden isometric subgraphs from the lemma.

(i.e. a shortest path) in  $G$  between  $u$  and  $v$  that lies entirely in  $H$ . In other words,  $d_H(u, v) = d_G(u, v)$  for any  $u, v \in V(H)$ , where  $d_G$  as usual denotes the shortest path distance in  $G$  (and similarly for  $d_H$ ). It is obvious that an isometric subgraph is also an induced subgraph.

**Lemma 1.** Let  $G$  be a graph such that  $S(u, v, w) = I(u, v) \cup I(v, w) \cup I(u, w)$  for all (not necessarily distinct)  $u, v, w \in V(G)$ . Then  $G$  does not contain the diamond,  $C_4$ , and  $C_t$ , for  $t \geq 6$  as an isometric subgraph. If, in addition, there exists an integer  $k \geq 4$  such that  $S(u_1, \dots, u_k) = \cup_{i \neq j} I(u_i, u_j)$  for all (not necessarily distinct)  $u_1, \dots, u_k \in V(G)$  then  $G$  does not contain an induced  $C_5$ .

**Proof.** For every graph from the list of forbidden subgraphs, one can find a triple  $u, v, w$ , such that  $I(u, w)$  is not a subset of  $S(u, v, w)$  and so the condition from the lemma is violated. In Fig. 1 the appropriate  $u, v, w$  are depicted. Then  $S(u, v, w)$  in each of the subgraphs is a path, while  $I(u, v) \cup I(v, w) \cup I(u, w)$  contains all vertices of the depicted subgraph. Vertex  $x$  in each of these subgraphs belongs to  $I(u, w)$ , and is not in  $S(u, v, w)$ . This is clear in the case of  $K_4 - e$  and  $C_4$ . Since  $d_{C_6}(\{u, v, w\}) = 3$  and as  $x$  is not adjacent with either  $v$  or  $w$ , it is not on a Steiner tree for  $u, v$ , and  $w$ . Consider now the vertices  $u, v$ , and  $w$  in the  $C_7$  of Fig. 1. We show that  $d_G(\{u, v, w\}) = 4$ . Since these vertices form an independent set,  $d_G(\{u, v, w\}) \geq 3$ . However, no vertex is adjacent with all three of these vertices; otherwise, the  $C_7$  in Fig. 1 is not an isometric subgraph. So  $d_G(\{u, v, w\}) \geq 4$ . Since the  $u - v$  path of length 2 of the  $C_7$  in Fig. 1 followed by the  $v - w$  path of length 2 is a tree with 4 edges that contains the vertices  $u, v$ , and  $w$ , it follows that  $d_G(\{u, v, w\}) \leq 4$ . One can now argue as for the  $C_6$  that there is no vertex that is adjacent with both  $v$  and  $w$  and at least one of  $u$  or  $x$ , otherwise, the  $C_7$  of Fig. 1 is not an isometric subgraph.

Similarly, when  $G$  contains an isometric subgraph  $C_{2k} : u_1 u_2 \dots u_{2k} u_1$  or  $C_{2k+1} : u_1 u_2 \dots u_{2k+1} u_1$ , for  $k \geq 4$ , we take  $u = u_1, v = u_k, w = u_{k+1}$ , resp.  $w = u_{k+2}$  in the odd case, and verify for  $x = u_{2k}$ , resp.  $x = u_{2k+1}$ , that  $x$  is not in  $S(u, v, w)$ , while clearly it is in  $I(u, w) = S(u, v, w)$ . Since cycles are isometric one can infer that any Steiner tree that contains  $u, v, w$  and  $x$  is of size at least  $k + 2$  (for odd cycles  $k + 3$ ), while there exists a Steiner tree for  $\{u, v, w\}$  of size  $k + 1$  (for odd cycles  $k + 2$ ). The details of the verification are left to the reader.

For the second part, consider the case when the  $k$ -Steiner interval satisfies the union property where  $k > 3$ . Note that the diamond, and the cycles that cannot be isometric in the case  $k = 3$  also cannot be isometric for  $k > 3$ . Let  $u_1, \dots, u_5$  be the vertices of an induced 5-cycle in  $G$ , ordered in the natural way. Note that an induced  $C_5$  is also isometric. Then  $S(u_1, \dots, u_1, u_2, u_3, u_4) = \{u_1, u_2, u_3, u_4\}$  while  $u_5 \in I(u_1, u_4)$ , which is a contradiction. Hence an isometric  $C_5$  is not possible in this case.  $\square$

## 2. Case $k > 3$

A block of a graph is a maximal connected subgraph without a cut vertex. A graph  $G$  is called a block graph if and only if every block of  $G$  is a clique. Note that block graphs are precisely chordal diamond-free graphs, hence they are also the graphs in which there are no induced diamonds and no isometric cycles of length more than 3, cf. [4].

**Lemma 2.** Let  $G$  be a block graph,  $U = \{u_1, u_2, \dots, u_k\}$  a multiset of vertices of  $V(G)$  and  $C = \{c \mid c \notin U \text{ is a cut vertex of } G \text{ which lies on some shortest } u_i, u_j\text{-path}\}$ . Then  $S(u_1, u_2, \dots, u_k) = U \cup C$ .

**Proof.** Let  $G, U$  and  $C$  be as in the statement of the lemma. Let  $x \in U \cup C$ . If  $x = u_i$  for some  $i$  then obviously  $x \in S(u_1, u_2, \dots, u_k)$ . Now, suppose  $x$  is a cut vertex of  $G$  which lies on some shortest  $u_i, u_j$ -path. Since  $u_i$  and  $u_j$  belong to different connected components of the graph  $G - x$ ,  $x$  lies on every Steiner tree for  $U$  and is thus in  $S(U)$ . Observe that we can find a Steiner tree containing exactly the vertices of  $U$  and  $C$ , thus we derive that  $d(U) = k + |C| - 1$ . From this we deduce that  $S(U)$  cannot include any additional vertex besides vertices of  $U$  and  $C$ , which completes the proof of the lemma.  $\square$

**Theorem 3.** Let  $G$  be a connected graph and  $k > 3$ . The following statements are equivalent:

- (i)  $G$  is a block graph,
- (ii) the  $k$ -Steiner interval on  $G$  satisfies (m),

- (iii) the  $k$ -Steiner interval on  $G$  satisfies (b2),
- (iv) the  $k$ -Steiner interval on  $G$  satisfies the union property.

**Proof.** (i)  $\Rightarrow$  (ii). Let  $G$  be a block graph. We prove that  $S(x_1, \dots, x_k) \subseteq S(u_1, \dots, u_k)$  for every  $x_1, x_2, \dots, x_k \in S(u_1, u_2, \dots, u_k)$ . Let  $a \in S(x_1, x_2, \dots, x_k)$ . By Lemma 2,  $a$  is either a vertex from  $\{x_1, x_2, \dots, x_k\}$ , or it is a cut vertex of  $G$  that lies on a shortest path between two vertices from  $\{x_1, x_2, \dots, x_k\}$ .

If  $a = x_i$ , for some  $i$ , then  $a \in S(u_1, \dots, u_k)$  as desired, since  $x_i \in S(u_1, u_2, \dots, u_k)$ . So assume that  $a$  is a cut vertex lying on a shortest path between  $x_i$  and  $x_j$  for some  $i, j$ . We have the following three cases concerning  $x_i$  and  $x_j$  with respect to  $U = \{u_1, u_2, \dots, u_k\}$ .

**Case 1.**  $x_i = u_k$ , and  $x_j = u_\ell$ , for some  $u_k, u_\ell \in U$ .

In this case  $a \in S(u_1, u_2, \dots, u_k)$  by Lemma 2.

**Case 2.** One of  $x_i$  and  $x_j$  is in  $U$  and the other is not.

We can choose the notation such that  $x_i = u_m$  for some  $u_m \in U$  and  $x_j \neq u_\ell$ , for any  $u_\ell \in U$ . Hence  $a$  lies on a shortest path between  $x_j$  and  $u_m$ . Since  $x_j \neq u_\ell$  for any  $\ell$ ,  $x_j$  is a cut vertex of  $G$  which lies on some shortest  $u_i, u_j$ -path. Hence  $G - x_j$  is a graph with at least two connected components, where  $u_i$  is in one and  $u_j$  in another. Assume without loss of generality that  $u_m$  lies in some other component as  $u_i$ . Then we have  $d(u_i, u_m) = d(u_i, x_j) + d(x_j, u_m) = d(u_i, x_j) + d(x_j, a) + d(a, u_m) = d(u_i, a) + d(a, u_m)$ , since  $a$  lies between  $x_j$  and  $u_m$  and is thus in the same component as  $u_m$ . We derive that  $a$  is a cut vertex of  $G$  that lies on a shortest  $u_i, u_m$ -path and hence  $a \in S(u_1, u_2, \dots, u_k)$ .

**Case 3.** Both  $x_i$  and  $x_j$  are not from  $U$ .

Since  $x_i, x_j \notin U$ , both  $x_i$  and  $x_j$  are cut vertices lying on some  $u_p, u_r$  and  $u_q, u_s$  shortest path, respectively, which may be distinct or the same. If both  $x_i$  and  $x_j$  are vertices lying on the same shortest path, say on the shortest path between  $u_p$  and  $u_r$ , then  $a$  is also a cut vertex lying on the same shortest path and the theorem is proved. Otherwise note that  $x_i$  and  $x_j$  are not in the same block (since  $a$  is on the shortest  $x_i, x_j$ -path). Let  $u_p$  be in a different connected component of  $G - x_i$  as  $x_j$ , and let  $u_q$  be in a different connected component of  $G - x_j$  as  $x_i$ . Then the vertices  $a, x_i$ , and  $x_j$  lie on a shortest  $u_p, u_q$ -path, which proves  $a \in S(u_1, u_2, \dots, u_k)$ .

(ii)  $\Rightarrow$  (iii) As noted in Section 1, this implication always holds.

(iii)  $\Rightarrow$  (i). Suppose the  $k$ -Steiner interval satisfies the (b2) axiom in  $G$  and suppose  $G$  is not a block graph. Then clearly  $G$  contains an induced diamond or it contains an isometric  $C_t$ ,  $t \geq 4$ . First suppose  $G$  has an induced  $K_4 - e$  with vertices  $u, w$  having degree 2 with respect to the diamond, and  $v$  is another vertex of the diamond. Then  $S(u, \dots, u, w)$  is not a subset of  $S(u, \dots, u, v, w)$ . Hence  $S$  does not satisfy (b2), a contradiction. Now suppose  $G$  has an isometric  $C_t$  with vertices  $u_1, \dots, u_t$ ,  $t \geq 4$ . If  $t = 2m$  then  $S(u_1, u_1, \dots, u_1, u_{m+1})$  is not a subset of  $S(u_2, u_1, \dots, u_1, u_{m+1})$ , and if  $t = 2m + 1$ , then  $S(u_1, \dots, u_1, u_{m+1}, u_{m+2})$  is not a subset of  $S(u_2, u_1, \dots, u_1, u_{m+1}, u_{m+2})$ ; both are contradictions.

(i)  $\Rightarrow$  (iv) By Lemma 2,  $S(U) = U \cup C$ . Note that  $U \cup C$  equals  $\bigcup_{i \neq j} I(u_i, u_j)$ , since every vertex in  $I(u_i, u_j) \setminus \{u_i, u_j\}$  for  $i \neq j$  is a cut vertex and thus either in  $U$  or in  $C$ .

(iv)  $\Rightarrow$  (i) Suppose  $S(U) = \bigcup_{i \neq j} I(u_i, u_j)$  for every multiset  $U = \{u_1, u_2, \dots, u_k\}$  of vertices in  $G$  (and thus also for the 3-Steiner interval in which there are at most 3 distinct vertices in  $U$ ). Hence by Lemma 1 we derive that every block in  $G$  contains no isometric cycles of length greater than 3. Hence a shortest cycle in every block is a triangle, and since there are no induced diamonds, and no greater isometric cycles (by the same lemma again) one easily infers that each block is a clique. Thus  $G$  is a block graph.  $\square$

### 3. Case $k = 3$

#### 3.1. Graphs in which the 3-Steiner interval satisfies the union property

We begin this section with the structural result about 3-Steiner intervals in the class of geodetic graphs. A graph is called *geodetic* if there is a unique shortest path (alias geodesic) between every pair of vertices. These graphs were considered by several authors, see for instance [3,12,19,23,25].

Obvious examples of graphs, in which there is a unique geodesic between every pair of vertices, are odd cycles, trees and complete graphs, but we shall come across several others during our study. Note that a graph is geodetic if and only if any of its blocks is geodetic [25]. Now, take a triple of distinct vertices  $u, v, w$ , and consider the three geodesics between pairs of the triple. By the structure of these graphs, the geodesics themselves form the corresponding intervals between the pairs. Let  $u'$  be the last vertex that is common to the  $u - v$  and  $u - w$  geodesics (possibly  $u' = u$ ), and similarly we define  $v'$  and  $w'$ . Then there is a block  $B$  containing  $u', v'$  and  $w'$ . Now, we are ready to state the following (straightforward) result which has a similar role as Lemma 2, where Steiner intervals were studied in block graphs.

**Lemma 4.** Let  $G$  be a geodetic graph (i.e. a graph in which there is a unique geodesic between every pair of vertices). Let  $u, v$ , and  $w$  be arbitrary distinct vertices of  $G$ , let  $u', v'$  and  $w'$  be defined as above. Then  $S(u, v, w) = I(u, u') \cup I(v, v') \cup I(w, w') \cup S(u', v', w')$ .

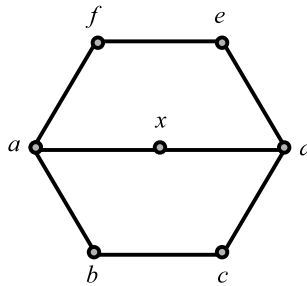


Fig. 2. An induced subgraph in Case 2.

It is easy to prove the lemma, so we will skip the proof. Note that in the above formula  $I(u, u')$  is simply the geodesic between  $u$  and  $u'$ . Although the above result is not interesting if there is only one block, it trivially holds for 2-connected graphs. It also holds if two of the vertices from the triple  $u, v, w$  coincide.

For  $n, m \geq 3$  let  $C_{m,n}$  denote the graph obtained from the cycles  $C_m$  and  $C_n$ , by amalgamating them along an edge of each cycle. For example,  $C_{3,3}$  is the diamond and  $C_{3,4}$  is the house.

**Theorem 5.** Let  $G$  be a connected graph. Then  $S(u, v, w) = I(u, v) \cup I(v, w) \cup I(u, w)$  for all  $u, v, w \in V(G)$  if and only if every block in  $G$  is either a complete graph or  $C_5$ .

**Proof.** First, let  $G$  be a connected graph such that  $S(u, v, w) = I(u, v) \cup I(v, w) \cup I(u, w)$  for all  $u, v, w \in V(G)$ . Let  $B$  be a block in  $G$  different from  $K_2$ . Therefore  $B$  contains a cycle. Let  $C$  be a shortest cycle contained in  $B$ . Note that  $C$  is then also isometric in  $G$ . Hence by Lemma 1 it is only possible that  $C$  is isomorphic to  $C_3$  or  $C_5$ , hence we distinguish these two cases.

**Case 1.**  $C \cong C_3$ . Let  $C'$  be a clique (maximal complete subgraph) that contains  $C$ . If  $C'$  equals  $B$  then the theorem follows. Otherwise, let  $D$  be a shortest cycle in  $B$  which contains only two vertices of  $C'$ , say  $a$  and  $b$ . This cycle is again isometric. Hence, by Lemma 1, it is isomorphic to  $C_3$  or  $C_5$ . Suppose  $D$  is isomorphic to  $C_3$ . Then let  $a, b, d$  be its vertices, and let  $c$  be a vertex of  $C'$  that is not adjacent to  $d$ . Such a vertex exists, since  $C'$  is a maximal complete subgraph. Then  $a, b, c, d$  induce the diamond which is a contradiction with Lemma 1. Hence we may assume that  $D$  is isomorphic to  $C_5$ .

Let  $c$  be a vertex of  $C'$ , distinct from  $a$  and  $b$ . We show that the union of  $D$  and  $c$  induces a subgraph, isomorphic to  $C_{3,5}$ . First it is clear that the neighbors  $a'$  and  $b'$  on  $D$  of  $a$  and  $b$ , respectively, are not adjacent to  $c$ , otherwise as above we obtain a diamond. Also, the remaining vertex  $x$  of  $D$  cannot be adjacent to  $c$ , because that would imply that  $C_5$  is not the shortest cycle, containing two vertices from  $C'$  (namely  $x, c, b$  and  $b'$  would form a  $C_4$ ). Thus  $V(D) \cup \{c\}$  induces a  $C_{3,5}$ . Also  $V(D) \cup \{c\}$  induces an isometric subgraph, except in the case when there is a path of length 2 between  $x$  and  $c$ . Hence there are two possibilities for induced subgraphs that can appear, see Fig. 5 where both graphs are depicted. First, in the graph on the left side of Fig. 5, the Steiner interval  $S(a, c, x)$  is of size four, and so one easily finds that  $b$  and  $b'$  cannot be in  $S(a, c, x)$ . On the other hand  $b, b' \in I(x, c)$ , which yield a contradiction with the hypothesis. For the graph on the right side of Fig. 5, the Steiner interval  $S(a', b', c)$  contains all vertices of the subgraph, while  $y \notin I(a', b') \cup I(a', c) \cup I(b', c)$ .

We conclude that in the case  $C \cong C_3$ , the block  $B$  is isomorphic to the complete graph  $C'$ .

**Case 2.**  $C \cong C_5$ . If  $C$  is equal to  $B$ , the theorem follows. Otherwise, let  $D$  be a shortest cycle in  $B$  which contains at least two vertices of  $C$ , and is not equal to  $C$ . Note that it could contain also three vertices of  $C$ . This cycle is also isometric, hence the subgraph induced by its vertices can only be isomorphic to  $C_5$  by Lemma 1 and minimality of  $C$ .

First let us consider the case when  $D$  shares three vertices with  $C$ . Then the graph in Fig. 2 is a subgraph of  $B$ . Note that this is an induced subgraph in  $G$ . Indeed, there can be no edges between two vertices of  $C$  (respectively  $D$ ), since they are isometric cycles, and also there can be no edges between  $b$  (or  $c$ ) and  $e$  (or  $f$ ) because we would get a shorter cycle than  $C_5$ . Note, as above, that the size of the Steiner tree for  $a, e$  and  $c$  is 5, and so all vertices are contained in  $S(a, e, c)$ . On the other hand,  $x$  is not in any of the three intervals, since we have  $d(a, e) = d(a, c) = d(c, e) = 2$ , and the subgraph is induced.

In the rest of the proof of this direction, we consider the case when  $D$  shares exactly two (adjacent) vertices with  $C$ . Then  $\langle C \cup D \rangle$  contains a  $C_8$  where a pair of antipodal vertices of the  $C_8$  is adjacent. One can quickly check that only one other edge in this subgraph is possible (using the fact that  $C$  and  $D$  are isometric and that  $C_5$  is a shortest cycle), and this is the edge between another pair of antipodal vertices that are not adjacent to the vertices of  $C \cap D$ .

Suppose that this edge exists, see Fig. 3, where the graph that appears as an induced subgraph is depicted. Let  $U = \{u, v, w\}$ . Then the Steiner distance  $d(U)$  of  $U$  is 4 (it cannot be 3, since, say  $u$  and  $v$  cannot have another common neighbor), hence  $S(u, v, w)$  contains all vertices of this subgraph. Now,  $x$  is not in  $I(u, v) \cup I(v, w) \cup I(u, w)$ , otherwise the subgraph of Fig. 3 is not induced.

Now, suppose that the union of  $C$  and  $D$  yields an induced subgraph (see the graph  $C_{5,5}$  as depicted in Fig. 4(a)). Let  $u, v$  and  $w$  be vertices of this subgraph as depicted in this figure. Let  $U = \{u, v, w\}$ . Note that  $d(U) \leq 4$ . On the other hand, since the subgraph is induced,  $d(U) \geq 3$ . If  $d(U) = 3$ , then  $u$  and  $v$  have a common neighbor  $z$ , or  $u$  and  $w$  have a common neighbor  $y$  ( $z \neq y$  since there are no triangles). In the first case we obtain as a subgraph in  $G$  the graph of Fig. 4(b). This is also an induced subgraph in  $G$  which can be easily checked. Now, vertices  $a, b, c, d, u, z$  and  $v$  induce a subgraph isomorphic



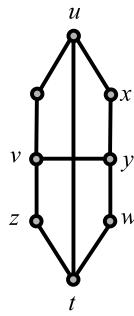


Fig. 3. An induced subgraph in Case 2.

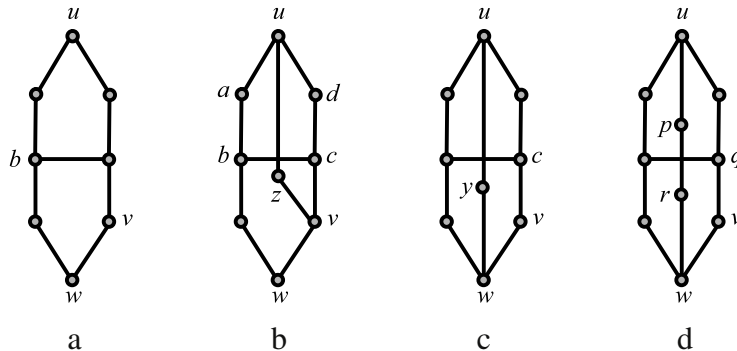


Fig. 4. Induced subgraphs in Case 2.

to the graph of Fig. 2, which we have shown is not possible. Suppose  $u$  and  $w$  have a common neighbor  $y$ , as shown in Fig. 4(c). Since we have shown that  $u$  and  $v$  have no common neighbor,  $d_G(u, v) \geq 3$ . So  $c \in I(u, v)$ . But  $c \notin S(u, v, w)$ , a contradiction. Thus  $d(U) = 4$ .

Now, consider the distance in  $G$  between  $u$  and  $w$ . It cannot be equal to 1 or 2 as we have already observed. If  $d_G(u, w) = 3$ , then, since the graph of Fig. 4(b) is not possible,  $G$  must contain the graph of Fig. 4(d) as a subgraph. One can check that this subgraph is induced otherwise one of the earlier situations arises. In this case the vertices  $p$  and  $r$  belong to  $I(u, w)$  but they cannot belong to the Steiner interval  $S(u, w, q)$ . In the last case, when  $d_G(u, w) = 4$  (see the graph of Fig. 4(a)),  $b \in I(u, w)$  but  $b \notin S(u, v, w)$ , a final contradiction, and the proof of this direction is completed.

For the converse, let  $G$  be a graph in which every block is either a complete graph or  $C_5$ . Then it is easy to see that for every two vertices  $u, v \in V(G)$  there is a unique shortest path connecting  $u$  and  $v$ . So  $G$  is a geodetic graph. Consider three vertices  $u, v, w \in V(G)$ , and let  $u', v', w'$  be as defined prior to Lemma 4. Then by this lemma,  $S(u, v, w)$  is the union of the vertices on the  $u, u'$ -geodesic, those on the  $v, v'$ -geodesic, those on the  $w, w'$ -geodesic and those in  $S(u', v', w')$ . The structure of the Steiner interval  $S(u', v', w')$  depends on whether  $B$  is a complete graph or  $C_5$  and on whether some (or all) of  $u', v'$  and  $w'$  coincide, and can be easily analyzed. (For instance, if  $B$  is  $C_5$ , then  $S(u', v', w')$  is a path, if vertices of the triple lie on some  $P_3$  of the cycle, otherwise  $S(u', v', w')$  is the entire cycle.) In all cases  $S(u', v', w') = I(u', v') \cup I(v', w') \cup I(u', w')$  which implies  $S(u, v, w) = I(u, v) \cup I(v, w) \cup I(u, w)$ , as desired.  $\square$

### 3.2. Graphs in which the 3-Steiner interval satisfies (m)

In this section we aim to characterize the graphs in which the 3-Steiner interval satisfies the (m) axiom (respectively, the (b2) axiom). It turns out that the class of graphs with (b2) is strictly larger than the class of graphs with (m). We will characterize the latter class of graphs, and at the same time look at the former (for which we will present some partial observations that will lead to a conjecture about their structure).

**Lemma 6.** Let  $G$  be a graph in which the 3-Steiner interval satisfies (b2) axiom. Then  $G$  does not contain the diamond,  $C_4$ , and  $C_t$ , for  $t \geq 6$  as an isometric subgraph.

**Proof.** We refer to Fig. 1 and use similar arguments as in Lemma 1. The condition (b2) is not satisfied if  $G$  contains an induced subgraph isomorphic to a  $C_4$  or a diamond as labeled in Fig. 1, since  $S(u, u, w) = I(u, w)$  is not a subset of  $S(v, u, w)$ . For cycles  $C_t$ ,  $t \geq 6$  we can use the same notation as in Lemma 1. In the same way we derive that  $S(u, u, w) = I(u, w)$  is not a subset of  $S(u, v, w)$  which concludes the proof.  $\square$

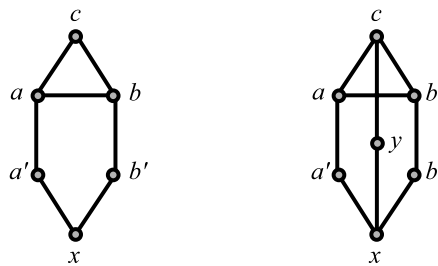
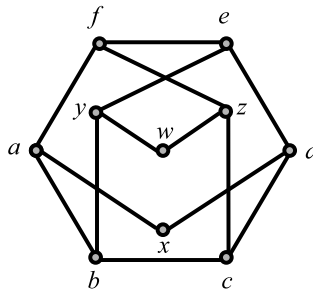
Fig. 5. Graphs  $C_{3,5}$  and  $M_3$ .

Fig. 6. A graph from the proof of Lemma 9.

Graphs  $C_{3,5}$  and  $M_3$ , depicted in Fig. 5, will play an important role in what follows. We will also use the notion of a  $g$ -convex set in a graph  $G$ , which is a set of vertices  $W \subseteq V(G)$  such that for any  $u, v \in W$ ,  $I(u, v) \subseteq W$ . Clearly, an intersection of  $g$ -convex sets in a graph  $G$  is also a  $g$ -convex set, and the smallest  $g$ -convex set that contains a set  $U \subset V(G)$  is called the convex hull of  $U$ .

**Lemma 7.** Let  $G$  be a graph in which the 3-Steiner interval satisfies condition (b2) and let  $H$  be a subgraph of  $G$ , isomorphic to  $C_{3,5}$ . Then the convex hull of  $H$  is either the complete graph, or  $H$  is an induced subgraph in  $G$  and its convex hull is isomorphic to  $M_3$ .

**Proof.** Let  $G$  be a graph in which the 3-Steiner interval satisfies the (b2) axiom. Suppose that there is a subgraph  $H$  in  $G$  isomorphic to  $C_{3,5}$ . Suppose that the convex hull of  $H$  is not a clique. Then it is not hard to see that  $H$  is induced in  $G$  otherwise we obtain an isometric  $C_4$  or diamond which is impossible by Lemma 6. Note that the distance between every two vertices of  $H$  is at most two in  $H$ , except the distance between  $c$  and  $x$  which is 3 in  $H$ . We show that  $H$  cannot be isometric in  $G$ . Let  $U = \{a, c, x\}$ . First observe that  $d(U) = 3$  and that  $b \notin S(a, c, x)$ . On the other hand  $b \in S(c, c, x)$ , which is a contradiction to the assumption that (b2) is satisfied in  $G$ . Hence  $d_G(c, x) = 2$ , thus there exists a common neighbor  $y$  of  $c$  and  $x$ . The resulting graph (isomorphic to  $M_3$ ) is  $g$ -convex in  $G$ , since if we connect any two vertices in  $M_3$  that are at distance 2 by a path of length 2 whose internal vertex is not in  $M_3$ , we obtain a forbidden  $C_4$  or diamond.  $\square$

We follow with another property of graphs with (b2).

**Lemma 8.** Let  $G$  be a graph in which the 3-Steiner interval satisfies the condition (b2) and let a 6-cycle  $C$  be an induced subgraph of  $G$ . Then every pair of antipodal vertices in  $C$  has a common neighbor (and all three neighbors are pairwise different).

**Proof.** Let  $C : abcdefa$  be a 6-cycle in  $G$ . Suppose (without loss of generality) that  $a$  and  $d$  do not have a common neighbor. Then  $d_G(a, d) = 3$ , hence  $S(a, a, d)$  contains all vertices of  $C$ . But  $S(a, b, d)$  does not contain  $f$ . Hence the triple  $a, b, d$  does not satisfy the condition (b2), a contradiction. Hence every pair of antipodal vertices in  $C$  has a common neighbor not on  $C$ . Obviously all of them are pairwise different, otherwise we obtain an isometric  $C_4$  or a diamond as induced subgraph which is impossible by Lemma 6.  $\square$

In the case of (m), Lemma 8 can be further strengthened.

**Lemma 9.** Let  $G$  be a graph in which the 3-Steiner interval satisfies condition (m). Then  $G$  does not contain an induced 6-cycle.

**Proof.** Suppose that  $C$  is an induced 6-cycle in  $G$  with vertices  $a, b, c, d, e, f$ . By Lemma 8 (we can use it since if a graph satisfies condition (m) it satisfies also the condition (b2))  $a$  and  $d$  have a common neighbor  $x$ ,  $b$  and  $e$  have a common neighbor  $y$ , and  $c$  and  $f$  have a common neighbor  $z$ . It is easy to check that the 6-cycle  $bczfeyb$  is induced in  $G$ , otherwise a contradiction to Lemma 6 arises. Hence by Lemma 8,  $y$  and  $z$  have a common neighbor  $w$ , see Fig. 6.

Let  $U = \{a, c, e\}$ . We now show that  $d(U) = 4$ . It is easy to see that  $d(U) \neq 2$  since  $C$  is induced, and  $d(U) \neq 3$  otherwise  $a, c$ , and  $e$  would have a common neighbor which would lead to an isometric  $C_4$  or diamond, contrary to Lemma 6. From

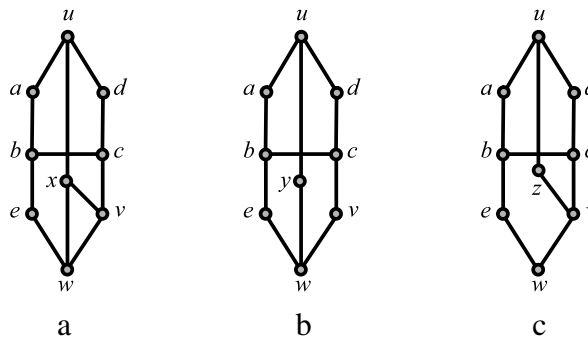


Fig. 7. Cases from Lemma 10.

this it is easy to see that  $S(a, c, e)$  includes every vertex depicted in Fig. 6 except the vertex  $w$  (otherwise  $w$  would have to be adjacent to at least one of the vertices  $a, c, e$ , but this again eventually leads to a contradiction with Lemma 6). Let  $W = \{y, z, e\}$ . Observe that  $d(W) \geq 3$ , since  $z$  cannot be adjacent to  $y$  or  $e$ . Also, since  $eywz$  is the path that contains all vertices of  $W$ , we infer  $d(W) = 3$ . Hence  $S(y, z, e)$  is not contained in  $S(a, c, e)$  which contradicts the assumption that  $G$  satisfies the condition (m).  $\square$

**Lemma 10.** Let  $G$  be a graph in which the 3-Steiner interval satisfies condition (m). Then  $G$  does not contain the graph  $C_{5,5}$  as an induced subgraph (see Fig. 4(a)).

**Proof.** Suppose that the graph isomorphic to the graph in Fig. 4(a) is an induced subgraph of  $G$ . Let  $u, v$  and  $w$  be vertices of this graph as labeled in Fig. 4(a), and let  $U = \{u, v, w\}$ . Note that  $d(U) \leq 4$ . Since the  $C_{5,5}$  is induced,  $d(U) \geq 3$ .

If  $d(U) = 3$ , then  $u, v, w$  have a common neighbor  $x$  or only  $u$  and  $w$  have a common neighbor  $y$  or only  $u$  and  $v$  have a common neighbor  $z$ , see Fig. 7.

In the first case when the graph in Fig. 7(a) appears, we can (using Lemma 6) easily check that either this graph is induced or the edge  $ax$  exists. If the graph in Fig. 7(a) is an induced subgraph of  $G$ , then also the 6-cycle  $xwebaux$  is induced, a contradiction with Lemma 9. If the edge  $ax$  exists, then by Lemma 7 there exist vertices  $s$  and  $t$ , a common neighbor of  $u$  and  $e$ , and  $d$  and  $w$ , respectively (note that  $s$  and  $t$  cannot coincide since otherwise we would obtain a forbidden isometric 4-cycle  $xutwx$ ). But then the 6-cycle  $sudtwes$  is induced. Indeed, the following edges are not possible:

- $sw$  (and  $ut$  by symmetry), otherwise  $swxus$  is an isometric 4-cycle, since the edge  $sx$  cannot exist,
- $sd$  (and  $et$  by symmetry), otherwise the 6-cycle  $sdcvwe$  would be induced ( $s$  cannot be adjacent to any of  $c, v, w$ , otherwise we obtain either an isometric 4-cycle or diamond),
- $st$ , otherwise  $s, t, d, u$  induce a  $C_4$ ,
- $uw, ue, dw$  and  $de$ , since, by the assumption, the graph in Fig. 4(a) is induced.

This implies a contradiction with Lemma 9.

Now suppose that the graph from Fig. 7(b) appears (where  $y$  and  $v$  are not adjacent, otherwise we obtain a graph isomorphic to the graph 7(a)). Note that  $y$  and  $c$  cannot be adjacent either. If  $yd$  is not an edge, then vertices  $y, u, d, c, v, w$  would induce a 6-cycle, a contradiction to Lemma 9. But if there is the edge  $yd$ , we get the graph from Fig. 7(a) for which we have already proved that it is not possible.

The last case to consider is the case when the graph from Fig. 7(c) appears. In this case (since we assume that  $z$  and  $w$  are not adjacent) two situations are possible: either the graph from Fig. 7(c) is an induced subgraph of  $G$  or there is an edge  $az$ . In the former case, also  $uabcvzu$  is an induced 6-cycle, and if  $az$  exists then  $azvwbea$  is an induced 6-cycle, a contradiction to Lemma 9.

We derive that  $d(U) > 3$ , and so  $d(U) = 4$ . Hence  $d_G(u, w) \geq 3$ . If  $d_G(u, w) = 3$  this implies that the graph from Fig. 4(d) is a subgraph of  $G$  (since the graph from Fig. 7(c) is forbidden), in which vertices  $p$  and  $r$  belong to  $I(u, w) = S(u, u, w)$  but they cannot belong to  $S(u, w, q)$ . In the case when  $d_G(u, w) = 4$ , the graph in Fig. 4(a) is a subgraph of  $G$ , with  $b \in I(u, w) = S(u, u, w)$  but  $b \notin S(u, v, w)$ , the final contradiction.  $\square$

Now we introduce graphs  $M_n$ ,  $n > 1$ . They can be constructed from the complete graph  $K_n$  with vertices  $x_1, \dots, x_n$  and a star  $K_{1,n}$  with  $x$  as its center and  $y_1, \dots, y_n$  as its leaves, by adding an edge between  $y_i$  and  $x_i$  for  $i = 1, \dots, n$ . It is straightforward to verify that in these graphs, called  $M_n$ , the 3-Steiner interval satisfies (m) and hence (b2), see Fig. 8.

**Theorem 11.** Let  $G$  be a connected graph. Then the 3-Steiner interval satisfies condition (m) in  $G$  if and only if each block of  $G$  is either a complete graph, or a graph isomorphic to  $M_n$  for  $n \geq 2$ .

**Proof.** Let  $G$  be a connected graph in which the 3-Steiner interval satisfies the condition (m), that is, for any  $u_1, u_2, u_3 \in V(G)$  and any  $x_1, x_2, x_3 \in S(u_1, u_2, u_3)$ , we have  $S(x_1, x_2, x_3) \subseteq S(u_1, u_2, u_3)$ .



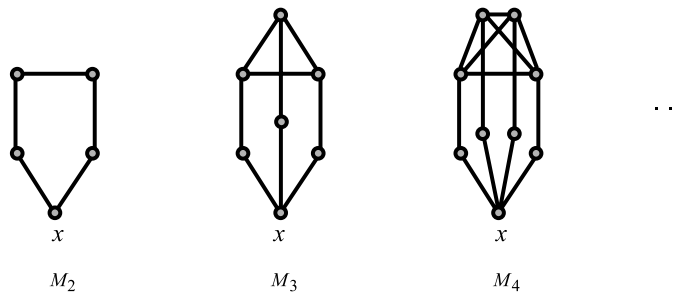


Fig. 8. Graphs with condition (m):  $M_2, M_3, M_4, \dots$

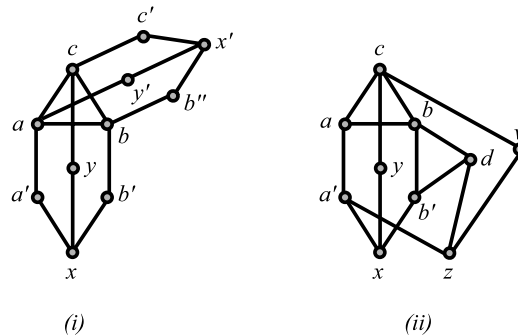


Fig. 9. Graphs in the proof of Theorem 11.

By Lemma 6  $G$  does not contain any  $C_4$ , diamond or  $C_t$ ,  $t \geq 6$ , as an isometric subgraph. Let  $C$  be a smallest cycle  $C$  in a block  $B$  of  $G$ . Then  $C$  must be a triangle or a 5-cycle.

**Case 1.**  $C \cong C_3$ . The proof of this case is the same as the proof of Case 1 of Theorem 5 up to the case where we encounter the graph  $C_{3,5}$  from Fig. 5. In this case, we use Lemma 7 and derive that  $M_3$  as labeled in Fig. 5 appears as an induced subgraph of  $G$  where  $C = abca$ . Let  $C'$  be a maximal clique that contains  $C$ . If there is a vertex  $d \in C'$  different from  $a, b, c$ , then again by Lemma 7, there must be a vertex  $y'$  adjacent to both  $d$  and  $x$ . If  $y = y'$ , then the subgraph induced by  $b, c, d$  and  $y$  is the diamond which is forbidden by Lemma 6. So  $y \neq y'$ . By repeating the same argument for every vertex  $c'_i$  in  $C'$ , there exists a distinct vertex  $y'_i \neq y$  in  $D$  which is a common neighbor of  $c'_i$  and  $x$ . Thus we obtain the graph  $M_n$  (see Fig. 8) where  $n$  is the size of the clique  $C'$ .

Now we prove that the resulting graph  $M_n$  coincides with the entire block  $B$ . Suppose to the contrary that there are some other vertices in  $B$ . Hence, let  $Q$  be a shortest path between two vertices of  $M_n$  such that no vertex of  $Q$  (except its end vertices) is a vertex of  $M_n$ . Because of the symmetry in  $M_n$  it is enough to look at the graph  $M_3$  and distinguish between the following possibilities:

- (i)  $Q$  is a path between vertices  $b$  and  $c$ . By Lemma 6,  $Q$  has length 4. Let its interior vertices be  $b'', x'$  and  $c'$  as labeled in Fig. 9(i). We claim that the set of vertices  $S = \{c, b, b'', x, y, b', x', c'\}$  induces a graph isomorphic to  $C_{5,5}$ . First note that by Lemma 7 there exists  $y'$  that is a common neighbor of  $a$  and  $x'$ . Again by Lemma 7 we only have to check the nonexistence of edges starting in  $c', x'$ , or  $b''$  and ending in  $y, x$ , or  $b'$ . However all these edges cannot exist by the minimality of  $Q$ . Now we arrive at a contradiction since by Lemma 10,  $C_{5,5}$  cannot be induced in  $G$ .
- (ii)  $Q$  is a path between vertices  $b$  and  $b'$ . By Lemma 6 and minimality,  $Q$  has length 2 or 4. Suppose first that its length is 2, and let  $d$  be a common neighbor of  $b$  and  $b'$ . By Lemma 7 there exists a vertex  $z$  that is a common neighbor of  $d$  and  $a'$  and again by the same lemma there exists  $v$ , a common neighbor of  $c$  and  $z$ , see the graph in Fig. 9(ii). Note that  $ycvza'xy$  is a 6-cycle, and let  $H$  be the subgraph induced by its vertices. If any one of the edges  $yz, xv$ , and  $xz$  is in  $H$ , it is a part of an induced  $C_4$  or diamond which is forbidden. By Lemma 7 also all other edges of  $H$  are forbidden with the possible exception of  $yv$ . Suppose  $yv \in E(H)$ . But then  $yzdb'xy$  is another 6-cycle which is induced contrary to Lemma 9. Indeed, all other edges in  $H$  are forbidden either by Lemma 7 or they force an induced  $C_4$  or a diamond (we leave the details to the reader).
- Suppose now that the length of  $Q$  is 4. Note that vertices of  $Q \cup \{a, a', x\}$  induce a subgraph that contains  $C_{5,5}$  as spanning subgraph. By Lemma 10,  $G$  cannot contain  $C_{5,5}$  as an induced subgraph. Thus there must be an edge from  $a'$  to a central vertex of  $Q$ . But this contradicts the minimality of  $Q$ .
- (iii)  $Q$  is a path between vertices  $b'$  and  $x$ . Again the length of  $Q$  can only be 2 or 4 by the same reasons as above. Suppose  $Q$  has length 2. Let  $d$  be the central vertex of  $Q$ . By Lemma 7 there exists a vertex  $z$  that is a common neighbor of  $a$  and  $d$ . Then  $acyxda$  is a 6-cycle, and let  $H$  be the subgraph induced by its vertices. By Lemma 9,  $H$  is not induced, and using

**Lemma 7** we infer that only the edges  $yd$ ,  $zc$ ,  $cd$ , and  $zy$  could be added between vertices of  $H$ . However, the first two edges force a diamond, while  $cd$  and  $zy$  force an induced  $C_4$  or diamond, a contradiction in each case.

Suppose now that the length of  $Q$  is 4. Then vertices of  $V(Q) \cup \{y, c, b\}$  induce a subgraph that is isomorphic to  $C_{5,5}$ , which cannot be induced by **Lemma 10**. Hence there must be an edge from  $c$  to the central vertex of  $Q$ , contrary to the minimality of  $Q$ .

- (iv)  $Q$  is a path between vertices  $b$  and  $x$ . By **Lemma 6** and minimality of  $Q$  the length of  $Q$  is 3. Let  $x'$  and  $b''$  be the neighbors of  $x$  and  $b$ , respectively, on  $Q$ . Then  $a'abb''x'xa$  is a 6-cycle in  $G$  which is either induced (which is a contradiction to **Lemma 9**), or it is not induced (in which case we arrive at a contradiction to the minimality of  $Q$ ). Hence such a path cannot exist.
- (v)  $Q$  is a path between vertices  $b'$  and  $c$ . In this case the length of  $Q$  must again be 3 (by **Lemma 6** and minimality of  $Q$ ). Let  $c'$  and  $b''$  be the neighbors of  $c$  and  $b'$ , respectively, on  $Q$ . We derive that  $cc'b''b'xyc$  is a 6-cycle in  $G$ , and conclude similarly as in the previous case that this is not possible.
- (vi)  $Q$  is a path between vertices  $b'$  and  $a'$ . By the same reasoning as in case (iv) we find that the length of  $Q$  must be 3. Let  $a''$  and  $b''$  be the neighbors of  $a'$  and  $b'$ , respectively, on  $Q$ . Now  $aa'a''b''b'ba$  is a 6-cycle in  $G$  and we conclude similarly as in the previous two cases. Hence  $B$  is isomorphic to  $M_n$ .

**Case 2.**  $C \cong C_5$ . If  $C$  is equal to  $B$ , the theorem is proved. Otherwise, let  $D$  be a shortest cycle in  $B$  that contains at least two vertices of  $C$ , and is not equal to  $C$  (note that it might contain also three vertices of  $C$ ). This cycle is also isometric, hence it can only be isomorphic to  $C_5$  by **Lemma 6** and minimality of  $C$ .

First let us consider the case when  $D$  shares three vertices with  $C$ . Then the graph in **Fig. 2** is a subgraph of  $G$ . In addition, there can be no edges between two vertices of  $C$  (respectively  $D$ ), since  $C$  and  $D$  are isometric cycles. Also there can be no edges between  $b$  (or  $c$ ) and  $e$  (or  $f$ ), as labeled in **Fig. 2**, because this would imply that there is a shorter cycle in  $B$  as  $C_5$ . Hence the graph in **Fig. 2** is an isometric subgraph of  $G$ . But then  $abcdefa$  is an induced 6-cycle, a contradiction to **Lemma 9**.

The second and final case is that  $D$  shares exactly two (adjacent) vertices with  $C$ . Let  $H$  be the subgraph of  $G$  induced by  $V(C) \cup V(D)$ . As  $C_{5,5}$  cannot be an induced subgraph in  $G$  (by **Lemma 10**), we derive that the graph in **Fig. 3** is a spanning subgraph of  $H$ , and is in fact isomorphic to  $H$  (since no additional edges are possible between vertices in that figure). But then  $H$  contains the graph from **Fig. 2** as an induced subgraph, which in turn contains an induced 6-cycle, a contradiction with **Lemma 9**.

For the converse, we will use **Lemma 4**. Notably, if every block of  $G$  is a complete subgraph or isomorphic to  $M_n$  then there is a unique geodesic between every pair of vertices in  $G$  (i.e.  $G$  is a geodetic graph). We also know that if  $G$  is  $M_n$ , then the monotone condition is satisfied. Using **Lemma 4** and the definition prior to this lemma, for any triple of vertices  $u, v, w \in V(G)$  we have  $S(u, v, w) = I(u, u') \cup I(v, v') \cup I(w, w') \cup S(u', v', w')$ , where  $u', v', w'$  all lie in the common block  $B$ . Let  $x, y, z \in S(u, v, w)$ . If  $x, y, z$  all belong to either  $I(u, u')$ , or  $I(v, v')$  or  $I(w, w')$ , then  $S(x, y, z)$  will be a subset of  $I(u, u')$ , or  $I(v, v')$  or  $I(w, w')$  as the case may be, and hence  $S(x, y, z)$  is a subset of  $S(u, v, w)$ . If  $x, y, z \in S(u', v', w')$ , then  $S(x, y, z)$  is contained in the block  $B$  and is also a subset of  $S(u, v, w)$  since every block of  $G$  satisfies (m). If two of  $x, y, z$ , say  $x, y \in I(u, u')$  (without loss of generality we can assume that  $y \in I(x, u')$ ) and  $z \in I(v, v')$ , then  $S(x, y, z)$  consists of  $I(x, u') \cup I(z, v')$  and a subset of  $S(u', v', w')$  (which can be a proper subset or the whole of  $S(u', v', w')$ , depending upon the structure of  $S(u', v', w')$  and the nature of the vertices  $x, y, z$ ). In this case also  $S(x, y, z)$  is contained in  $S(u, v, w)$ . The cases when  $x \in I(u, u')$ ,  $y, z \in S(u', v', w')$  and  $x \in I(u, u')$ ,  $y \in I(v, v')$ ,  $z \in S(u', v', w')$  can be handled similarly as above, and are left to the reader. The final case is when all of  $x, y, z$  lie in different intervals, say  $x \in I(u, u')$ ,  $y \in I(v, v')$  and  $z \in I(w, w')$ . In this case, we have  $S(x, y, z) = I(x, u') \cup I(y, v') \cup I(z, w') \cup S(u', v', w')$ . Here also  $S(x, y, z) \subseteq S(u, v, w)$ . Thus the 3-Steiner interval satisfies condition (m) and the proof of the theorem is complete.  $\square$

#### 4. Concluding remarks

1. The most evident problem that arises from this paper is a structural characterization of graphs in which the 3-Steiner interval satisfies (b2) axiom. **Lemmas 6–8** already give a lot of information about the structure of these graphs. The main distinction with axiom (m) comes in **Lemma 9**, where it is proved that  $C_6$  cannot be an induced subgraph of graphs satisfying (m). In the case of (b2) the 6-cycle can be induced, and the graph from **Fig. 6** is a subgraph of the Petersen graph. It is straightforward to check that in the Petersen graph the 3-Steiner interval satisfies the (b2) axiom. Moreover, we think that this holds in all geodetic graphs with diameter 2, which is a much larger class of graphs (see Stemple [25] for a characterization of these graphs and more examples). We think that the converse could also be true, and state this as the following conjecture.

**Conjecture 12.** Let  $G$  be a connected graph. The 3-Steiner interval satisfies the condition (b2) in  $G$  if and only if each block of  $G$  is a geodetic graph with diameter at most 2.

2. The Steiner set in a graph  $G$  is defined as a set  $W \subseteq V(G)$  whose Steiner interval  $S(W)$  equals  $V(G)$ . As mentioned in the introduction, an erroneous statement from [6] encouraged some further investigation in the area. The statement was that every Steiner set is a geodetic set (i.e. a set  $W$  of vertices in a graph such that every vertex of a graph lies on a geodesic interval between two vertices from  $W$ ). However this is true in some graphs, but it is an open problem to determine a

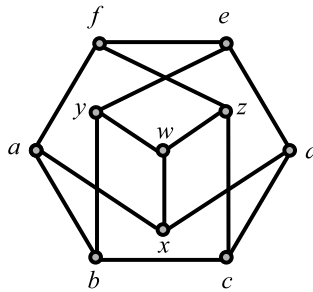


Fig. 10. The Petersen graph satisfies (b2), but not (m).

characterization for these graphs. A related (stronger) question could be more easy to solve (based on the results of this paper): In which graphs  $G$ , for every set  $W \subset V(G)$ , the Steiner interval  $S(W)$  is included in the union of geodesic intervals, that is

$$S(W) \subseteq \bigcup_{u_i, u_j \in W} I(u_i, u_j).$$

We could also restrict the question to  $k$ -Steiner intervals for particular integers  $k$ , where  $k > 2$ . Also, the reversed inclusion would be interesting: In which graphs  $G$ , for every set  $W \subset V(G)$ ,  $\bigcup_{u_i, u_j \in W} I(u_i, u_j) \subseteq S(W)$ ? Observe that, for  $|W| = 3$ , the  $M_n$ 's from Section 3 satisfy this property.

3. We can easily consider the (b1) axiom in the way the other conditions were handled in this paper. Let  $G$  be a graph on at least three vertices. For a complete graph  $G$ , we have  $S(u_1, u_2, \dots, u_k) = \{u_1, u_2, \dots, u_k\}$  and hence  $S$  satisfies (b1). On the other hand, if  $G$  is a connected graph on at least 3 vertices that is not complete, then there exist vertices  $u, v, w$  with  $v \neq u$  such that  $vu, uw \in E(G)$  but  $vw \notin E(G)$ , then  $S(u, v, \dots, v, w) = \{u, v, w\}$  and thus  $v \in S(u, v, \dots, v, w) = \{u, v, w\}$ , but we have  $u \in S(v, v, \dots, v, w)$ , hence (b1) is not satisfied.

**Remark 13.** The  $k$ -Steiner interval, where  $k > 2$ , of a connected graph  $G$  satisfies the (b1) axiom if and only if  $G$  is a complete graph.

The question remains if one can characterize the graphs in which the Steiner interval satisfies other betweenness axioms, as introduced in [15].

4. One could compare the Steiner intervals also with other natural binary intervals. In particular, the main question from this paper could be posed for monophonic (induced path) interval  $J$ . It was shown by Hernando et al. [10] that every Steiner set in a connected graph is a monophonic set — that is, vertices of a Steiner set  $W$  have the property that every vertex in  $G$  lies on an induced path between two vertices from  $W$ . However the following natural question is open: in which graphs do we have that for every set  $W$

$$S(W) = \bigcup_{u_i, u_j \in W} J(u_i, u_j)?$$

Furthermore, what is the relation of the resulting class with the house, hole, domino-free graphs? The latter class of graphs was studied also with respect to another type of convexity (namely  $m^3$  convexity), based on the so-called  $m^3$  intervals [7].

5. When  $W$  is a set instead of a multiset, then the classes of graphs, in which the  $k$ -Steiner interval for  $W$  has the union property, satisfies the monotone axiom (m), and satisfies betweenness axiom (b2), are in general not the same as what we have obtained in this paper. For example, it can be easily verified that in any cycle  $C_n$  the  $k$ -Steiner interval for  $k$ -sets satisfies the monotone axiom (m) and hence the (b2) axiom. Also there are graphs other than complete graphs in which the  $k$ -Steiner interval satisfies (b1) axiom; for instance the stars, the paws etc. Thus when no repetitions of the vertices are allowed, the corresponding problem for  $k$ -Steiner intervals is another interesting problem. However the class of graphs in which the 3-Steiner interval satisfies the union property is the same class as what we obtained for the union property in this paper. That is, Theorem 5 holds also for 3-Steiner intervals  $S(u, v, w)$  with distinct  $u, v, w$ .

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